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Essay Review

‘Having the Answers’: Writing the History of Mathematics in India

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Mathematics in India
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> A mathematician [‘calculator’] is to be known by eight qualities: swift work, deliberation, refutation, non-idleness, comprehension, concentration [or memory], inventiveness, and having the answers.
>
> Mahāvīra *Gaṇitasārasaṅgraha* 0.69

‘Like the fruit in one’s palm’, analogizes 12th-century Indian mathematician Bhāskara II (p. 199),[1] a sentiment which fittingly captures Kim Plofker’s work *Mathematics in India*—prized fruit indeed that has been embraced by the eagerly-waiting scholarly community and is fast establishing itself as both an essential and desirable addition to the hands of many researchers. Plofker surveys over 2500 years of one of the richest, most complex, and most fascinating mathematical traditions that in many ways contrasts markedly with other, more familiar, mathematical traditions. She aims to enrich and, where appropriate, challenge the various orthodoxies that exist about the discipline. Indeed, this rich tradition has often been characterized by but a few features—the decimal place value system, infinite power series, indeterminate equations—a picture which, as she shows, is staggeringly incomplete. Plofker addresses this discrepancy by presenting thorough, reliable, and reflective accounts that allow the sources to speak for themselves. In addition, she emphasizes the broader themes and characteristics of this tradition and its status and position in a wider context. In this way, Plofker’s work gives background, depth, and coherency to a tradition in desperate need of attention. Her accomplishment is masterful.

To do truly excellent work in a discipline in which so much is still significantly under-explored is not authoritatively to declare any conclusive statements. Rather, true scholarship is to set the finest example so far of the synthesis of many threads of prior contributions, and

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[1] Bhāskara’s point, in fact, was that mathematicians who remained unaware of broader workings of mathematics (and he had in mind ‘spherics’) and who could only carry out calculations, would not attain greatness.

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to build upon them with insight from the past and a view to the future. In this way, Plofker has illuminated a path for those she knows are following and her work is not only a significant achievement but also calls for further advancement in the field.

Mathematical indologists have challenges coming at them from every direction. Indeed, Plofker conveys a vivid sense of some of the difficulties in this field, providing us with a solid appreciation of the level of specialization and skills required to undertake this work. There is a multi-dimensionality in her approach. She brings multiple language fluency (including several arcane languages and scripts), paleographical facility, and codicological experience. In addition, she exhibits keen mathematical insight and expertise, teasing meaning out of often terse and obscure sources and relating them in the modern terms, a solid historical background needed to appreciate and contextualize these texts, ingenuity to consider the ways in which these practitioners developed and devised these relations in the first place, and a knowledge of other fields to highlight and explore the interrelationships, such as astronomy, astrology, and religious traditions, as well as the mathematical traditions of other cultures.

Mathematics was thoroughly integrated into many aspects of everyday life in India. As mid-9th-century mathematician Mahāvīra noted, this discipline manifested itself in 'all fields of practical or religious relevance, including love, commerce, music, theater, cooking medicine, building, prosody, rhetorical ornamentation, poetry, logical argument, grammar, astronomy, and cosmology' (p. 163) and the various elements of mathematics were likened in metaphor to an ocean, in the way that many parts make up the whole. Thus, the task Plofker faced was huge, and in addition to the challenges of scope and scale, there are the issues of interpretation and contextualization. Indeed, as she observes many of the details of the history of Indian mathematics have long resisted the attempts of mathematicians and historians to describe their origination and meaning. Challenges arise not only because of the presentation, medium, and format of these works, but also because of the facility and ingenuity with which the Indian mathematicians themselves generated and developed mathematical knowledge and combined their geometrical and numerical methods (p. 88). Plofker confronts these aspects head on, tackling the outstanding problems using mathematical considerations, historical perspectives, linguistic analysis, logic, and persistence.

Challenges abound from other aspects also. This field is utterly reliant on the preservation of primary sources and scholars are keenly aware of the fact that they are at the mercy of the circumstances which determine whether or not the key manuscripts have actually physically survived the vicissitudes of time. Furthermore, as Plofker notes, in a field where even basic chronological information is impossible to be sure of, this discipline is full of contradictory interpretations and accounts. She negotiates her way through this historiographical mélange expertly and respectfully, accounting for divergent views where appropriate and acting with the highest of academic professionalism. Much of her secondary scholarship narrative is contained in the footnotes; her historiographical commentary is subtle yet sharp. Highlights include Gary Tubb's penetrating insights into a mathematical double entendre (p. 183, fn. 24), the bewildering claims of Subhash Kak [2000] concerning deep numerical significances encoded in Vedic corpus (p. 35, fn. 25), the paranoia of John Bentley [1970] (pp. 115–116, fn. 78), the mathematically sophisticated yet flawed approach of Billard [1971] (p. 116, fns. 80–83), a re-evaluation of the generalizations of prominent scholars Datta and Singh [1962] (p. 60, fn. 41, and p. 279, fn. 1), a critical appraisal of al-Bīrūnī [Sachau, 1992] (p. 261, fn. 18), and the clarification of the term 'Vedic mathematics' (p. 16, fn. 7), among others.

Plofker also expertly manages the necessary and complicated task of speaking to multiple audiences. Inquiry into the history of science is always, by nature, an interdisciplinary affair, and the need to satisfy and speak to various audiences is essential. Material is considered and evaluated in different lights, and this work will satisfy the exacting standards of many disciplines, including mathematics, history, indological studies, philosophy, religious studies, linguistics, material culture, and textual criticism, to name a few.

Plofker has recently contributed a survey of Indian mathematics in an extended book chapter in Katz [2007]. Prior to that summary, various key works exploring this tradition have been of variable quality, but those that are commonly referred to include Colebrook [1817], Datta and Singh [1962], and Pingree's monumental multi-volumed census of practitioners [Pingree, 1970–81]. Sarasvati Amma focused on the discipline of geometry [Sarasvati Amma, 1999], and other studies which provide a thorough and reliable account of a single work, such as [Keller, 2006] and [Sakhya, 2009], are becoming more common.

Mathematics in India is organized largely chronologically. As well as the main content, this work also provides excellent reference information. The two large appendices provide a thorough overview of the Sanskrit language and reliable biographical data on just under 50 Indian mathematicians and astronomers, a resource otherwise not available in one place. An expansive and thorough bibliography contains around 300 entries. Almost 50 of these entries are by David Pingree which is a testament to the singular contribution this tireless scholar made to the field. In addition, Plofker includes two glossaries, one on transliterated technical terms (of Sanskrit and other languages) and the other (appearing early on in the main body of the book on pages 62–66) on core concepts in astronomy. She also provides pictures, tables, worked examples, quotes and diagrams, all of which enrich her analysis.

For these reasons, a certain amount of preliminary work needs to be covered before the mathematics itself is addressed. Plofker summarizes these details in her introductory chapter. Here, she highlights salient aspects from South Asian history, Sanskrit literature, and the exact sciences. She tackles some of the prejudices that scholars have had (and not just modern ones) about the epistemological status of mathematics. In particular, her discussion (p. 12) on the various criteria of validation in mathematics is critically important. Mathematics in India, unlike other societies, was not an epistemologically privileged discipline, and thinkers had recourse to a variety of means to validate their material, such as direct perception, inference, analogy, and authoritative testimony, among others. This point, which could arguably be expanded by Plofker, is a defining characteristic of the exact sciences in India and will be fascinating to philosophers and historians of science.

In India, various philosophical schools (such as the Nyāya and Sāṃkhya schools) explored the various means of acquiring valid knowledge, in many contexts, not just mathematics. This knowledge is captured by the Sanskrit term *pramā* and its derivative noun *pramāṇa*, corresponding roughly to 'proof' or 'authority' in English. The multi-valency of validation in this culture has been a chief source of criticism, but recently scholars have recognized that it points to something much richer about this tradition and deepens our perspective on the circumstances of theoretical thinking in human societies more generally.

Of further significance to the history of mathematics is Plofker's point about the lack of documentation to reveal identities and backgrounds of mathematical authors (pp. 11–12). This contrasts, as she points out, with other traditions, such as that of the Greeks, which have rich background biographical literature. Plofker notes that often in the absence of such biographical accounts, these gaps have been filled by pseudobiographical details which have little authenticity and have been of little help. In a way not considered before in this field, she draws scholarly attention to the ambient social history of India relative to the history of mathematics. Here, the influences are shown to be interactive. As they testify to the flourishing and character of mathematics, she considers aspects (primarily in Section 6.1) such as mathematicians and society, demography, the intellectual status of knowledge from one generation to the next, social and professional groups in India, mathematics education and training, and the role and access to education by women, audience, family background, class and society structure, patrons, the *pakṣa*s (pp. 70–72) or schools and various affiliations and loyalties towards particular systems to followers of these schools, as well as observant sociological details such as unusual caste flexibility among certain groups of mathematicians (p. 218).

With the scene set, Plofker considers the earliest traces of mathematical activity in Vedic India in the second millennium BCE, giving textual evidence to support the existence of nascent but undeniable mathematical ideas and development. She shows (pp. 14–15) that this mathematical activity was not only driven by practical demands but also from supra-utilitarian concerns. Her main focus is an analysis of the *Śulbasūtras*, a composition on ritual practice and she uses this opportunity to pause to reflect on what this ritual geometry adds to the understanding of this most ancient Indian mathematical thought (p. 26). When covering the Vedas and astronomy she gives a careful exposition of quantitative astronomy and related issues of dating Vedic texts (p. 28ff). In particular she raises the contentious methods of astrochronology (the application of modern principles of astronomy to historical records of celestial phenomena to establish a date), carefully articulating reasons why the many previous attempts to apply this practice to the Vedic texts have been, and always will be, unsuccessful. This has been a particularly controversial area, but Plofker navigates her way through this difficult terrain. Her rational discussion and scientific and astronomical reasoning convincingly support her concerns about previous interpretations, and will hopefully resonate, if even in some small degree, with those who insist otherwise and persist in this practice.

Indeed, as this topic shows, much of the subject matter of this book is embedded in the broader context of mathematical astronomy. Accordingly, various metrological conventions, astronomical systems, terms, and designations, must also be laid out so that readers can have an appreciation for these details in the texts. Working from basic assumptions, Plofker covers these aspects. In addition, despite giving the reader as much assistance as possible, she is wary of certain limitations that modern readers will have, commenting (p. 40) 'we have to resign ourselves to the fact that ancient mathematical texts in the service of sacred rites leave out a lot of the technical, methodological, and historical details we might desire.' In this context, Plofker also considers the intellectual connections and transmission between Vedic India and ancient Mesopotamia, and the links and inspiration that the Mesopotamian divination tradition seems to have provided to Indian authors.

Further early chapters cover aspects according to chronology and context of the mathematical traces in the early classical period (middle of the first millennium BCE). This time period was a transitional episode for Sanskrit itself, as it shifted from a primary language to a learned language (p. 43), and India saw the rise of significant religious traditions such as Buddhism and Jainism. The impact on mathematics of these two key developments is still difficult to establish fully although there are profound connections. Plofker explores the use of numbers and numerals and considers their representation in various different scripts and media, including some stunning visual evidence of inscriptions in caves (p. 44). She explores the emergence of decimal place-value numbers (p. 45), situating their arrival by at least the 7th century CE. She includes a delightful description of one of the 'verbal' notational systems, the so-called 'concrete number system' (*bhūtasaṅkhyā*).[2] Later in the work she gives details of the so-called *kaṭapayādi* system (p. 75) and the *varga* system which uses an alphanumeric coding to represent numbers (p. 73). However, Plofker passes the opportunity to talk more fully about *why* the development of these systems was so significant, linking its presence back to the consequences of practising mathematics in an oral environment. Long strings of numbers are often difficult to memorize, and so, by appeal to actual objects, both mundane and mythological, the concrete-number system made these long strings of numbers memorable. This very feature was the subject of criticism as early as the 11th-century, when Islamic scholar al-Bīrūnī deemed aspects of Indian mathematics muddled and confused as a result. Another highlight of particular importance in this period is the appearance of the trigonometric details generating various tabulated values of particular arcs, and the pertinent connections with Greece and the chord function counterpart. Mathematics at this time appeared in other disciplines as well (Section 3.3), not just texts in the exact sciences. In this light, Plofker considers mathematical aspects as seen in texts of grammar and prosody, as well as those that appear in religious texts (Section 3.4). In particular, as she notes, Jain conceptions of infinity still captivate scholars (p. 59).

Despite the occurrence of mathematical aspects in a variety of genres, at this time, by far the majority of mathematical literature was part of a wider discipline, that of *jyotiṣa*, the field combining astronomy, astrology, and calendrical concerns, and mathematical ideas were embedded in developed and complex astronomical contexts. Plofker fleshes out the necessary introductory details of geocentric astronomy for the reader. She introduces various textual formats including the *siddhānta* and the *karaṇa* and gives them comprehensive treatment. One aspect of her exposition is her balance between the old and the new. For example, not only does she give us a contemporary account of the *siddhānta* based on a modern impression of the extant sources, but also rounds her account with an early definition of this genre by the 10th-century scholar, Vaṭeśvara (p. 69). Of importance to mathematics with respect to the *siddhānta* genre are the following three areas (p. 69): (i) the determination of times, locations, and appearances of celestial phenomena; (ii) the explanation of the computational astronomy procedures in terms of the geometry of the spherical models underlying them; and (iii) the instruction in general mathematical knowledge.

Plofker makes some fascinating observations about mathematical diagrams and their status in this tradition. For instance, she comments (p. 98) 'eclipse computations are one of the very few instances in Sanskrit texts in which a user is instructed to draw a detailed mathematical diagram.' She does, however, draw our attention to a rare instance of a rudimentary eclipse diagram in a manuscript (p. 102). She also states that other features considered standard by many mathematical societies, such as letter-labeled points, are absent (p. 67). Plofker presents a detailed analysis of some ingenious formulae for trigonometric interpolation based on half-sums and half-differences (p. 106ff). Here, she could explore more deeply the emphasis on interpolation within this tradition—perhaps it was in some part a consequence of composing in an oral environment. Memory requirements

[2] Plofker has been somewhat deliberate about avoiding Sanskrit terms for concepts, a decision which is somewhat frustrating for an indologist familiar with Sanskrit, but understandable for the purpose of not overwhelming a more general readership. However, this is not without exception. For example, she calls the verbal notation discussed here, the *bhūtasaṅkhyā*, the 'concrete number system', but retains the Sanskrit term *kaṭapayādi* for another verbal system of numeration. This is almost certainly because the significance of this latter descriptive term would be lost when translated.

may encourage interpolation techniques, as they embody a priority of structure over particular details (which, of course, are harder to remember). Other aspects which are fascinating in this astronomical context are the interaction of geometry and arithmetic in astronomy (p. 110ff), the problem of origins and initial construction of parameters (p. 113) and a feature she identifies as 'computational positivism' (p. 120). Plofker argues that because of the variety of techniques that existed on the astronomical scene, both foreign and native, there resulted a certain flexibility of basic assumptions, and thinkers were not locked into any one fixed orientation. As a result, they could explore and create freely, largely unrestricted by cosmological or philosophical requirements. This theme is returned to later in the work (pp. 248–251) and will be critical to more general examinations of the development and improvement of astronomical models by scientific societies through history.

It was not until medieval times that works arose which were exclusively devoted to mathematics, and not just part of a wider programme, such as astronomy. Mathematical inquiry began to flourish as its own discipline. 'Mathematics', or *gaṇita*, now captured any type of computational or qualitative process (p. 121). To explore this new circumstance of mathematical inquiry, Plofker considers the work of Āryabhaṭa, Bhāskara, Brahmagupta, the Bakhshālī manuscript, and Mahāvīra in chronological order. She alerts her reader at the outset that as a result of the repetitive nature of many of these works, those that appear early have been given thorough treatment, and those further in, an ever reduced coverage, to avoid duplication. Her emphasis on the repetitiveness of content sits slightly uncomfortably with her conclusion that the mathematicians of this period 'diverge widely in their ideas even about the fundamental organization and presentation of mathematics' (p. 171) and she argues that these works are non-standard and individualistic. However, it is hard to fully appreciate this sentiment given the compounded descriptions of all of these works. With finesse, though, she combats the various 'facts' and 'extravagant claims' that have appeared about Indian mathematics of this period (p. 122) and carefully shows that they have no historically plausible validation. Photographic evidence of the Bakhshālī manuscript with accompanying transliteration and translation provides a satisfying sense of the structure of a mathematical text and casts light on the manifold steps involved in analyzing ancient mathematical documents. The reader can instantly appreciate the detail and complexities that one needs to account for, as well as enjoying the rewarding sense of really 'reading' an Indian mathematical text themselves.

Unlike the period preceding it, Plofker argues that by the mid-second millennium mathematical works were characterized by a standardization in their textual format. As a result, she reasons, mathematical knowledge had an increasingly uniform structure, and could be called 'canonical'. She identifies 12th-century mathematician Bhāskara II as emblematic of the mathematical activity in this period. Indeed, by mid-second millennium Bhāskara II's works were arguably the most widespread and are recognized even today as archetypal. It is also at this time that Plofker notes increased attention to the rationales behind mathematical processes and explanation. With standardization came aspirations on the part of mathematicians to go ever deeper (p. 198). The mathematical content of this chapter is gripping. The trigonometric identities Plofker uncovers (p. 205) will be fascinating to mathematicians. A reminder at this point for the reader might have been appropriate; for modern mathematicians the relations she illustrates are all essentially equivalent, but for those cultures without the same symbolic facility, nor generality of trigonometric functions, these identities remain distinct. Further details will also be of interest, including her ingenious recovery of Bhāskara's formula for computing terrestrial latitude (p. 207), Nārāyaṇa

Paṇḍita and his 'Net of Digits', the properties of cyclic quadrilaterals (p. 208),[3] magic squares, and series and their visual representation (p. 210). Again, her more general claim that the mathematics of this period is canonical (which is indubitably sound) is difficult to fully appreciate. For, as in the last chapter, her accounts highlight the nuances and novelties in each of the authors, and the fact that the texts of this period conformed increasingly to established standards is, as a result, not as evident.

What has rarely been offered in this field are more reflective appraisals on the state of the mathematical discipline in India. Here, Plofker offers insights of great value. She explores what Indian mathematicians thought about their craft and how they produced it (p. 210ff), particularly with respect to what they considered constituted *gaṇita*. Among the many key observations, she points out the fascinating fact that trigonometry *per se* was a special application in astronomy of geometry and, as the texts themselves reveal, was not considered part of more abstract mathematics at all. She characterizes the intellectual circumstances of mathematicians in India's history, stating '... we know of no major medieval Sanskrit mathematicians or astronomers who were also renowned as, say, poets or philosophers or authorities on dharma, although some displayed their non-mathematical erudition now and then in their technical works ... math remained mostly a technical speciality' (p. 212). She discusses the importance and centrality of orality and exposition (p. 212); her concise comments on this issue reveal that much further work needs to be done, as questions surrounding the practice of mathematics in an oral environment are probing, not just for Indian history, but other mathematical cultures as well. Plofker provides an excellent overview on the role of commentaries (p. 213ff) and insight into the related complex literary etiquette. The main incentive to write commentaries, she argues, was educational concerns. A concise oral exposition does not supply all the pedagogical needs in mathematics and thus commentaries functioned as a bridge. Plofker wraps up her discussion with a critical evaluation of the role of proof, observing that '...an individual mathematician's ingenuity rather than a formal methodology on which he had to rely for perceiving a mathematical fact in its different guises—verbal, numerical, symbolic, or geometric' meant that there was no responsibility on the part of mathematicians to prove their results (p. 215). This is a defining observation and will no doubt compel scholars to consider this issue further.

Plofker's chapter on the school of Mādhava in Kerala is a highlight. Her analysis is assisted by old commentaries, such as that of Śaṅṅkara. One thing that will astound modern scholars is the careful and conscious manipulation of long and complex expressions without formal symbolic expression by these mathematicians. The derivation of the circumference of a circle and the modern notation required to follow the derivation incites a level of deep admiration with respect to how these mathematicians worked. Plofker shows that the school of Mādhava produced more than just infinite series, and offers a broader account of the activity of this period. She complements her mathematical analysis with some pointed reflections on wider issues such as the disparity between mathematics and astronomy. The exploration of these intricate infinite series directs one to question their intended use and application. On this theme, Plofker concludes that, in fact, astronomers did not adopt these newly developed, highly accurate values in their computation—they

[3] Plofker comments 'when two adjacent sides are interchanged in a quadrilateral, a new quadrilateral is formed with (at least) one of its diagonals unchanged. This is true for cyclic quadrilaterals but not for arbitrary ones' (p. 208). This seems incorrect, for it is true for arbitrary ones as well.

simply did not require the precision to which they could compute π or evaluate sines and cosines in their astronomical computations. Rather, these relations seem to have been pursued entirely for mathematical curiosity and '...their chief significance appears to be not so much in their superior precision as in the insightful rationales by which they were derived' (p. 247).

In this light, she continues to explore the vital issue of the connections between astronomy and scientific methodology (Section 7.4). In particular, she investigates Nīlakaṇṭha and his ambition to explain astronomical investigations metaphysically and to synthesize *siddhānta* practices into a coherent mathematical model. Nīlakaṇṭha is renowned for an astronomical model that was effectively heliocentric (although it is unlikely that this was deliberate; see p. 251, fn. 47) and Plofker credits this achievement to the fluency and creativeness encouraged by this intellectual climate. Practitioners in India, she argues, were not bound by fixed basic assumptions (like the Greeks had been) to explain phenomena and could be imaginative when it came to considering how to adapt their astronomical models so that they might conform ever more closely to reality.

Given the issues raised by the appearance of infinite series in Kerala significantly earlier than their emergence in the 'west', many scholars have thought it right to raise the question of possible transmission to other mathematical cultures. In response to this issue, Plofker reflects upon the transmission between Kerala, Islamic cultures, and the European mathematical communities (pp. 251–253). She raises the open and probing question about transmission: 'the historiographic question thus raised is an interesting one: what are or should be the criteria for accepting a hypothesis of cross-cultural transmission as plausible, and are those criteria culturally dependent?' This point is relevant not just in this context but has a theoretical extension which can be applied to all cultures with significant knowledge transactions.

'Exchanges with Islam' is used by Plofker as a broadly descriptive term to cover a period of transmission with another rich mathematical culture, that of the Islamic Near East. Indian calculation (*ḥisāb al-hind*) was integral to this society and a defining part of Islamic activity. Readers are treated to an account of the passage of the decimal place value system through the Sassanian empire, 'zero' (p. 256), the trigonometry of sines and the engaging etymological trajectory from *jyā* to sine (p. 257), techniques of double false position (p. 259), al-Bīrūnī (p. 261) and his evaluation of mathematics in India, among other topics. She includes a discussion of instruments and astronomical tables where she keenly observes '...the Sanskrit bias in favor of orality evidently did not withstand the attractions of the labor saving...tables' (p. 274). She provides an illuminating section which compares Islamic mathematical works with their Sanskrit translations (p. 267), and details on Jayasiṃha's multilingual research team (p. 269). Despite these many interdependencies though, she observes that the Sanskrit and Arabic algebraic traditions are not much alike. There are strong connections, however, between successive or iterative approximation techniques which establishes transmission.

As the story enters the modern period, Plofker considers the continuity and changes to the mathematical discipline in India. She counters the prevailing claims that mathematical activity from the 14th to the 18th century was a period of deterioration. However, as India increased its contact with Europe after this time, she tracks the inevitable decline of the indigenous mathematical tradition, largely due to vernacularization, colonial rule, and modern globalization. As incentive to continued inquiry in this area, Plofker leaves us with many tantalizing questions (p. 296).

Plofker's coverage is impressive. But as in any work on such a large scale, there are inevitably going to be areas that various specialists will argue should have been covered or considered more deeply. For example, there is very little detail on the Chinese connection with the Indian mathematical tradition, despite the consideration of transmission to and from other mathematical societies, most notably Greece, Islam, and Europe. Plofker also focuses her efforts on the big names in mathematics, which means that lesser known mathematicians do not generally feature in her presentation, despite the fact that their work gives an important grounding to the tradition. Similarly, the later periods of Indian mathematical history have not been covered as thoroughly as earlier periods. The gradual erosion of the indigenous Indian mathematical tradition due to contact with Europe is no doubt fascinating, rich, and important for many reasons.

Also, while Plofker has provided the reader with helpful and ample cross referencing, the decision to follow a chronological presentation entails that there is less of an opportunity to collect and explore those themes which transcend a temporal ordering. I highlight a few of these for the interested reader, particularly as it is through these general themes that historians of mathematics from other fields can be better informed of the details and aspects of their culture of study and gain insight into similar themes in a contrasting intellectual environment.

One recurring theme which is of huge consequence for this tradition is the use of language. It is not simply the complexity of the Sanskrit[4] language (which Plofker covers thoroughly in Appendix A), but the fact that this intellectual tradition, like almost all such traditions in India, was carried out in an oral environment. This had a significant impact on the practice, format, articulation, and comprehension of mathematical content. The metrical requirements of verse had significant consequences on the technical vocabulary, for example. Furthermore, a practical orientation is suggested by much of the mathematical language. For instance, Brahmagupta's eight mathematical procedures include 'Mixtures', 'Excavations', 'Piles', 'Sawing', 'Heaps', and 'Shadows' (p. 141). On a deeper level, composer-mathematicians were often creative and even playful with their content through language. This can be seen in the use of puns and it is rare that non-Sanskritists have been able to share in these. Plofker embraces the opportunity to reveal the linguistic dexterity of these scholars, describing and explaining these where appropriate. For example, second degree indeterminate equations are captured by the Sanskrit term *varga-prakriti* by Brahmagupta, which translates literally as 'square-nature'; *varga* means square, *prakriti* means nature, and *kriti* means square. A captivating example of the clever use of language for double entendre culminates in Plofker's description of a verse (p. 183) in Bhāskara II's *Līlāvatī*. The effect of this passage is so impressive, it has been reproduced here entirely. The verse can be translated as:

> Those who keep in their throats the *Līlāvatī* having entirely accurate [arithmetic] procedures, illustrating elegant sentences, [whose] sections are adorned with excellent [rules for] reduction of fractions and multiplication and squaring…

which Plofker shows can be equally captured as:

> Those who clasp to their necks the beautiful one completely perfect in behaviour, enticing through the delight of [her] beautiful speech, [whose] limbs are adorned by the host of good qualities [associated] with good birth…

4 As a linguist, I often wished that I had the original Sanskrit text in front of me at times, although I know this would make the book uncontrollably large!

These key examples, along with many others in this book, combine to reveal something of the distinct and fascinating effects of the circumstances of oral transmission and its effects on practice.

Another significant theme is the issue of the pitch of the text, mathematical competency, and the effects on exposition with respect to these considerations. Throughout the history of Indian mathematics, authors have made a distinction between those results that are for the 'sharp witted' and those that are for the 'dull witted'. This distinction is intriguing, for it is often those results which are intended for the latter that contain the most thorough explanations and rationales, and are hence the more complete and revealing—quite the opposite to what we might expect. Those that are directed to the 'sharp' are typically less deep. This distinction and the way in which it manifests itself in texts also provides important insight into how practitioners esteemed their own work and pitched their results. For example the *karaṇa* format is for the 'stupid' (p. 108); Bhāskara notes that drawing a geometric diagram is 'to convince the unintelligent' and explicitly lists all the tabulated sines values 'for the slow-witted to perceive' (pp. 138–139). Bhāskara continues to shed light on this distinction elsewhere, commenting: 'but for increasing the intelligence of dull-witted ones like us, it has been explained by the wise in many different and easy rules' (p. 190). Fifteenth-century scholar Parameśvara wrote a work in just eight verses for the computation of the circumstances and details of eclipses, a topic he wrote about in nearly 100 verses elsewhere. He considers this highly compacted and terse edition to be for the 'dull-witted', despite the fact that it is so concise, it requires much insight to know what to do with the details. Along this theme, Plofker notes that it was not until late in Indian history (around the time of Mādhava) that rationales and explanations rose in epistemological status and became considered valid and important to include, and not just designed to assist the 'slow witted' (p. 247). This transition and the evolving exegetical tradition is fascinating and important to examine more closely.

Another theme which is nowhere treated on its own, but is critically important for mathematical practice is the issue of accurate versus approximate methods of computation, a distinction which was actively maintained by scholars. Instances abound in this work. Āryabhaṭa gives both a practical rule and a rule 'without remainder' for the volume of a sphere (p. 140). Brahmagupta (p. 144) describes approximate and accurate techniques to find areas of triangles and quadrilaterals. Mahāvīra gives an example of his wheel circle (p. 170) where the approximate and the accurate solution differ by multiplication by $3\sqrt{10}$. Bhāskara maintains this distinction in the geometry of circles (p. 190). Of mathematical interest are Mādhava's 'more' accurate value of π (p. 222) and his 'accurate' and 'very accurate' results for the circumference of a circle (p. 225). Even in the earliest mathematical texts certain rules are noted to have 'a difference' from the exact value (p. 21). In a related way, Plofker ingeniously probes the process of making a rule more 'accurate' in her account of the rule for establishing terrestrial latitude in Bhāskara's *Karaṇakutūhala*. She shows how practitioners may have tweaked simple and crude approximations to agree with a few known values, thus deriving a more accurate approximation. This distinction and the ways in which it pressed upon practitioners is significant. How could they tell which was better and what was the inspiration? Furthermore, in a discipline where the practical applications could not take advantage of a greater level of precision, what was the motivation for these developments?

For many historians of mathematics, not far from their individual scope of inquiry are 'algebraic' considerations. This theme comes up frequently throughout the book, algebra being often referred to as *bīja* (or seed) computation or sometimes 'unmanifest' mathematics (as opposed to the 'manifest' which is arithmetic; see for example Bhāskara (pp. 140, 192–193)). This visual orientation may be fruitful with respect to exploring how practitioners cognitively dealt with unknowns, particularly given the oral emphasis. Along this theme of the visibility element, it is interesting to consider those instances in which the unknowns are represented by colors; there is a certain significance in a very visual technique to depict something that is 'unmanifest'. Plofker has opted to 'translate' the types of reasoning with unknowns in the Indian tradition simply as algebra, perhaps losing some of the special qualities with which this aspect was considered by these people. The term 'algebra' may cloud the fact that these mathematicians did not have a fully refined symbolic abstract facility for example. She makes a tantalizing comment about the relations between geometry and algebra, noting that geometry was used by individuals to provide rationale to 'algebra'—geometry can demonstrate algebraic rules, but the reverse did not hold in their minds (p. 247). This complex and contemporary topic is worthy of further consideration.

To do justice to the mathematical details in this book is impossible. Some selected highlights that will excite mathematicians include: the detailed table (p. 27) of *Śulbasūtra* constants including embedded values of π, some of the very earliest decimal place value inscriptions (p. 45) and their later developments (pp. 255–256), the first surviving exemplar of an Indian sine table in the *Pañcasiddhāntikā* (p. 51), ingenious approximations to 2^n in early texts of prosody (p. 55) and the related computation of r^n (p. 165) by Mahāvīra by replacing multiplication with equivalent operations of addition, the worked description of an algorithm for retrieving the square root of 341,056 by Āryabhaṭa which Plofker annotates with her quirky 'oops start over' (p. 124) upon failure of the iteration, Brahmagupta's 'arithmetic of zero' (p. 151) and the calculation of factors of 0/0 being useful in astronomy (p. 198). Mathematicians will no doubt be amused by the following encouragement from Brahmagupta: 'whoever computes the answer within a year is a mathematician' (p. 154), which, paired with his paean of the profession, 'in the assemblies of the people he will destroy the brilliance of [other] astronomers as the sun [destroys that] of the stars' (pp. 100–101), will no doubt cause modern mathematicians to chuckle! Other features include an account of second-degree indeterminate equations (pp. 154–156), mathematical notation as a sort of proto-symbolic syncopated algebra (Section 5.2.1), the quantification of very small, infinitesimal like qualities (p. 163) in Mahāvīra, and Bhāskara II's ingenious demonstration of the computation of the area of a sphere by subdivisions (p. 199) as well as his general rule for $\sin(A + B)$ and related expressions. Mathematicians will relish in the fine details of the efforts of the school of Mādhava, including the numerical value for π (p. 221), the derivation of the circumference of the circle by means of polygons (p. 222), the Mādhava–Gregory series for arctangent (p. 230) and the Mādhava–Taylor series approximations for sine and cosine (p. 235). In addition there is also the most often quoted anecdote about Indian mathematics: Plofker gives us the real story of the Līlāvatī (p. 270) from the mouth of Abū al-Fayḍ Fayḍī, a renowned court scholar of the Mughal emperor Akbar in 1587. Scholars will be relieved to know that they have been telling this one correctly!

There is no doubt that with this book Plofker has contributed something of inestimable value to the field. Her accomplishment is propaedeutic to further needed exploration. She champions a new approach to the history of Indian mathematics and her dedication to the project is inspiring. Her scholarship will appeal to a wide audience, but at the same time retains a level of thoroughness, fastidiousness, and breadth that will make it an invaluable addition to the collections of active researchers. Her approach will serve as a paradigm and the momentum this book creates will prove irresistible to scholars to rejuvenate and re-evaluate the field in similar ways. This work is the culmination of many years of exposure to sources, expert and dedicated study of these texts, and scholarly maturity. Because of this, it is destined to become a long-standing classic.[5]

Of the eight qualities Mahāvīra listed as attributes of a mathematician (see the opening epigram), Plofker certainly epitomizes them all, with perhaps the exception of one! Her expediency, thoughtfulness, correctiveness, diligence, expertise, focus, and mathematical creativity will guide the discipline into the future. As for the last, that of 'having the answers', perhaps no one will ever be able to claim this completely. While Plofker tackles and in many cases solves a multitude of problems associated with this discipline, as she herself notes, there are still many mysteries and unanswered questions relating to these texts and those that wrote them. These indeed remain, with Plofker's guidance, to delight and dazzle future generations of scholars, just as their authors had intended.

References

Bentley, J., 1970. A Historical View of the Hindu Astronomy. Repr., Biblio Verlag, Osnabrück.

Billard, R., 1971. L'astronomie Indienne: Investigation des Textes Sanskrits et des Données Numériques. École Française d'extrême-orient, Paris.

Colebrook, H.T., 1817. Algebra, with Arithmetic and Mensuration, from the Sanskrit of Brahmegupta and Bhascara. J. Murray, London.

Datta, B., Singh, A.N., 1962. History of Hindu Mathematics, 2 vols. Repr., Asia Publishing House, Bombay.

Kak, S., 2000. The birth and development of Indian astronomy. In: Seline, H. (Ed.), Astronomy Across Cultures: The History of Non-Western Astronomy. Kluwer, Dordrecht.

Katz, V., 2007. The Mathematics of Egypt, Mesopotamia, China, India, and Islam. A Sourcebook. Princeton University Press, Princeton.

Keller, A., 2006. Expounding the Mathematical Seed, 2 vols. Birkhäuser, Basel [Reviewed by C. Montelle, Hist. Math. 35 (2008) 47–50].

Pingree, D., 1970–1981. Census of the Exact Sciences in Sanskrit. 4 vols. Amer. Phil. Soc., Philadelphia.

Sachau, E., 1992. Alberuni's India, 2 vols. Repr., Munshiram Manoharlal, New Delhi.

Sakhya (S.R. Sarma, T. Kusuba, T. Hayashi, M. Yano), 2009. Gaṇitasārakaumudī: The Moonlight of the Essence of Mathematics by Ṭhakkura Pherū. Ajay Kumar Jain, Manohar, New Delhi.

Sarasvati Amma, T.A., 1999. Geometry in Ancient and Medieval India, 2nd rev. ed. Motilal Barnasidass, Delhi.

[5] As far as errata are concerned, the manuscript is generally very clean. A few details this reviewer noticed were:

p. 125: b^2 should be b^3.

p. 130: 'sight' should perhaps be 'side'.

p. 219: The date '1393' should be '1398' which was the first attested eclipse observation by Parameśvara.

p. 246: vocalic l should perhaps be with a dot underneath.

p. 212: section 6.4.2 'middle of the second millennium' should be 'middle of the *first* millennium'.